Maths for Computing
Assignment 2 Solutions

1. *(3 marks)* Prove that if n is a perfect square, then $n + 2$ is not a perfect square.
Solution: Simper solutions also exist but I am going with the following.
We will prove it by contradiction. Suppose $n + 2$ is a perfect square.
We divide the rest of the proof in two cases (i) $n = 2k$, i.e., n is even and (ii) $n = 2k + 1$, i.e., n is odd.

Case 1: $n = 2k$
Since both n and $n + 2$ are perfect squares, there must exist integers l and m such that $n = l^2$ and $n + 2 = m^2$. This implies that l^2 and m^2 are even and hence l and m are also even. Let a and b two integers such that $l = 2a, m = 2b$.

Now, replace $n = 2k$ in $n = l^2$ and $n + 2 = m^2$ and add both the equations. We get,

$$\begin{aligned} 4k + 2 &= l^2 + m^2 \\ 2(2k + 1) &= 4a^2 + 4b^2 \\ 2k + 1 &= 2a^2 + 2b^2 \end{aligned}$$

Since one side is odd and other is even, we have reached a contradiction. Hence, $n + 2$ cannot be a perfect square.

Case 2: $n = 2k + 1$
Since both n and $n + 2$ are perfect squares, there must exist integers l and m such that $n = l^2$ and $n + 2 = m^2$. This implies that l^2 and m^2 are odd and hence l and m are also odd. Let a and b two integers such that $l = 2a + 1, m = 2b + 1$.

Now, replace $n = 2k + 1$ in $n = l^2$ and $n + 2 = m^2$ and add both the equations. We get,

$$\begin{aligned} 4k + 4 &= l^2 + m^2 \\ 4k + 4 &= (2a + 1)^2 + (2b + 1)^2 \\ 4k + 4 &= 4a^2 + 1 + 4a + 4b^2 + 1 + 4b \\ 4k + 2 &= 4a^2 + 4a + 4b^2 + 4b \\ 2k + 1 &= 2(a^2 + a + b^2 + b) \end{aligned}$$

Since one side is odd and other is even, we have reached a contradiction. Hence, $n + 2$ cannot be a perfect square.

2. *(3 marks)* Prove that if a and b integers and $a^2 + b^2$ is even, then $a + b$ is even.

Solution: If $a^2 + b^2$ is even, then there exists an integer k such that

$$a^2 + b^2 = 2k$$
$$a^2 + b^2 + 2ab = 2k + 2ab$$
$$(a + b)^2 = 2(k + ab)$$

Thus, $(a + b)^2$ is also an even number. And we have shown that if x^2 is even, then x is also even. Hence, $a + b$ is an even number.

3. *(4 marks)* Prove that for every $n \in \mathbb{Z}, 4$ does not divide $(n^2 + 2)$.

Solution: We divide the proof into two cases.

Case 1: n is even

Let k be an integer such that $n = 2k$. Then, $n^2 + 2 = 4k^2 + 2$. And $(4k^2 + 2) \% 4 = 2$. Hence, 4 does not divide $n^2 + 2$ in this case.

Case 2: n is odd

Let k be an integer such that $n = 2k + 1$. Then, $n^2 + 2 = 4k^2 + 4k + 3$. And $(4k^2 + 4k + 3) \% 4 = 3$. Hence, 4 does not divide $n^2 + 2$ in this case as well.

4. *(5 marks)* Suppose $a, b, c \in \mathbb{Z}$. Prove that if $a^2 + b^2 = c^2$, then at least one of a and b must be even.

Solution: Suppose both a and b are odd. Then there must exist integers k and l, such that $a = 2k + 1$ and $b = 2l + 1$.

Replace these values in $a^2 + b^2 = c^2$. We get,

$$(2k + 1)^2 + (2l + 1)^2 = c^2$$
$$4k^2 + 1 + 4k + 4l^2 + 1 + 4l = c^2$$
$$4(k^2 + l^2) + 4(k + l) + 2 = c^2$$
$$2(2(k^2 + l^2) + 2(k + l) + 1) = c^2$$

This proves that c^2 is an even number and hence c is also an even number. Let $c = 2m$, for some integer m. Replace $c = 2m$ in above equation. We get,

$$2(2(k^2 + l^2) + 2(k + l) + 1) = (2m)^2$$
$$2(2(k^2 + l^2) + 2(k + l) + 1) = 4m^2$$
$$2(k^2 + l^2) + 2(k + l) + 1 = 2m^2$$
$$2(k^2 + l^2 + k + l) + 1 = 2m^2$$

Which is not possible because one side is odd but other side is even. Thus, we have reached a contradiction and therefore, at least one out of a or b must be even.

5. *(5 marks)* Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

Solution: First observe that if x and y are integer solutions to this equation, then $|x|$ and $|y|$ will also be an integer solution to the equation. So, showing that the equation has no non-negative integers x and y as a solution is enough.

Clearly, $y \geq 2$ is not possible because that implies $5y^2 \geq 20 \implies 2x^2 + 5y^2 \geq 20 + 2x^2$ > 14. Similarly, $x \geq 3$ is also not possible. Thus, the only possible values of x are $\{0,1,2\}$ and of y are $\{0,1\}$. You can simply put these values in the equation and show that none of the pairs form a solution.

6. *(7 marks)* Prove that $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$, for every positive integer n.

Solution: *Basis Step:* For $i = 1$, $\frac{1}{1^2} \leq 2 - \frac{1}{1}$.

Inductive Step: Assume $P(k)$ and prove $P(k+1)$.

Since $P(k)$ is true, we can say

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$$
$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Now we can simply prove that $2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$.

$$\begin{aligned}
2 - \frac{1}{k} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k+1} \\
-\frac{1}{k} + \frac{1}{(k+1)^2} &\leq -\frac{1}{k+1} \\
\frac{1}{k+1} + \frac{1}{(k+1)^2} &\leq \frac{1}{k} \\
\frac{k+2}{(k+1)^2} &\leq \frac{1}{k} \\
k^2 + 2k &\leq (k+1)^2
\end{aligned}$$

The last inequality is trivially true. Although we started from the inequality we wanted to prove, we can also start from $k^2 + 2k \leq (k+1)^2$ and do the modifications in reverse order to prove the original inequality.

7. *(7 marks)* Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no

matter how you split the piles, the sum of the products computed at each step equals $n(n-1)/2$.
Solution: We use strong induction to prove it.
Basis Step: For $n = 1$ and $n = 2$, the statement is trivially true as for $n = 1$ there is no splitting and for $n = 2$, the only split will result in 1 which is equal to $2(2-1)/2$.

Inductive Step: Assuming $P(1), P(2), \ldots, P(n-1)$, we will prove $P(n)$.

Suppose we split a pile of n stones into r size and $n-r$ size piles. Then, the sum of products will be $r(n-r)$ + sum of products when you further break r size and $n-r$ size piles. From inductive hypothesis, we know that for r size and $n-r$ size piles, the sum of the product will always be $r(r-1)/2$ and $(n-r)(n-r-1)/2$. Therefore, the sum of the products while breaking n size pile will be $r(n-r)+r(r-1)/2+(n-r)(n-r-1)/2$, which can be simplified to $n(n-1)/2$.

8. *(8 marks)* At a tennis tournament, there were 2^n participants, where n is a positive integer, and any two of them played against each other exactly one time. Prove that we can find $n+1$ players that can form a line in which everybody has defeated all the players who are behind him in the line.
Solution:
Basis Step: For $n = 1$, there will be two players and the line can be formed by loser of the only match followed by the winner of that match.
Inductive Step: Assume $P(k)$ and prove $P(k+1)$.
Consider the winner of the tournament of 2^{k+1} many players. We claim that winner must have won at least 2^k matches. Why? Suppose winner has won less than 2^k or at most $2^k - 1$ matches, then everyone else must also have at most $2^k - 1$ matches. The total number of matches won by all the players is equal to the total number of matches happened because there are no ties.

The total number of matches won by everyone is at most $2^{k+1} . (2^k - 1)$. Now we will prove that this number is less than $\binom{2^{k+1}}{2}$, i.e., the number of matches held in the tournament. Hence, a contradiction.

$$2^{k+1} . (2^k - 1) < \binom{2^{k+1}}{2}$$

$$2^{2k+1} - 2^{k+1} < \frac{2^{k+1} . (2^{k+1} - 1)}{2}$$

$$2^{2k+2} - 2^{k+2} < 2^{2k+2} - 2^{k+1}$$

$$2^{k+1} < 2^{k+2}$$

So now take the set of 2^k players who were beaten by the winner. From the inductive hypothesis, we can say that among these players we can pick $k + 1$ players and line them up so that everyone has defeated all the players behind him in the line. Now, we can simply add the winner in front of the line to form a desired line of $k + 2$ players.

9. *(10 marks)* Prove the inequality between the geometric mean and the arithmetic mean using induction, that is, prove that if $a_1, a_2, \ldots, a_n$ are non-negative numbers, then

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

Solution: *Basis Step:* For $n = 1$, $\sqrt[1]{a_1} \leq \frac{a_1}{1}$ is trivially true.
For $n = 2$, $\sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2}$ can also be proven true easily.

Inductive Step: Proving $P(k) \rightarrow P(k+1)$ is difficult. But notice that proving $P(k) \rightarrow P(2k)$ for $k \geq 1$ and $P(k) \rightarrow P(k-1)$ for $k \geq 2$ is sufficient for us.

Proving $P(k) \rightarrow P(2k)$:

$$\begin{aligned}
\sqrt[2k]{a_1 a_2 \ldots a_{2k}} &\leq \sqrt{\sqrt[k]{a_1 a_2 \ldots a_k}\sqrt[k]{a_{k+1} a_{k+2} \ldots a_{2k}}} \\
&\leq \frac{\sqrt[k]{a_1 a_2 \ldots a_k} + \sqrt[k]{a_{k+1} a_{k+2} \ldots a_{2k}}}{2} && \because \sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2} \\
&\leq \frac{\frac{a_1 + a_2 + \ldots + a_k}{k} + \frac{a_{k+1} + a_{k+2} + \ldots + a_{2k}}{k}}{2} && \text{using} P(k) \\
&\leq \frac{a_1 + a_2 + \ldots + a_{2k}}{2k}
\end{aligned}$$

Proving $P(k) \rightarrow P(k-1)$:

$$\begin{aligned}
\frac{a_1 + a_2 + \ldots + a_k}{k} &\geq \sqrt[k]{a_1 a_2 \ldots a_k} \\
\frac{a_1 + a_2 + \ldots + \frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1}}{k} &\geq \sqrt[k]{a_1 a_2 \ldots a_{k-1} \frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1}} && \text{replacing } a_k \text{ with an appropriate value} \\
\frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1} &\geq \sqrt[k]{a_1 a_2 \ldots a_{k-1} \frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1}} \\
\left(\frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1}\right)^k &\geq a_1 a_2 \ldots a_{k-1} \frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1} \\
\left(\frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1}\right)^{k-1} &\geq a_1 a_2 \ldots a_{k-1} \\
\frac{a_1 + a_2 + \ldots + a_{k-1}}{k-1} &\geq \sqrt[k-1]{a_1 a_2 \ldots a_{k-1}}
\end{aligned}$$